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CALCULATION OF POWER FLOW BETWEEN COUPLED OSCILLATORS†

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A perturbation method is developed for calculating the statistics of the energy transfer process between weakly coupled oscillators. The method is used to calculate first-order approximations for (a) the mean value, and (b) the spectral density of the power flow between two stiffness coupled oscillators under white noise random excitation. The result for mean power flow is identical with that obtained by Lyon and Maidanik (1) using an *ad hoc* method of linearization. The present method is, however, more general in the sense that it allows more complicated statistics to be evaluated (for instance, the spectrum of the energy transmission process), and applies to cases with narrow-band as well as broad-band excitation. The present method also allows more accurate results to be obtained by calculating second and higher order approximations.

Among other possible applications, the method looks promising as an additional tool for the study of noise transmission in structures.

INTRODUCTION

In the last few years there has been an increasing interest in the statistical behaviour of sets of coupled oscillators. Their response may explain the performance of many apparently quite different physical systems. For instance, chemical reaction rates may be controlled by the rate of energy flow between the vibrational modes of molecules. The occurrence of biological rhythms may be due to a periodic cycling of energy between coupled oscillator systems, also at the molecular level. Wave interactions in fluids may be explained by the different rates of inter-modal energy transfer which occur. An important engineering application is the transmission of noise energy between coupled structures. An understanding of this process is important in the search for means of reducing acoustic noise, and also in controlling fatigue damage in structures subjected to random loading.

In this paper an approximate perturbation method is developed for calculating the statistics of the energy transmission process between weakly coupled oscillators. Although the method can be applied to any weakly coupled systems, it is developed here for systems with conservative "stiffness" coupling terms [see equation (2)]. The analysis follows closely previous work by the author devoted to the study of nonlinearly coupled oscillators (2). The results are applied to study the motion, under random excitation, of the two stiffness coupled oscillators shown in Figure 1. Since this two-degree-of-freedom system is so simple, the exact solution for the mean rate of energy transmission between the two oscillators is available for comparison with the approximate answer, and an estimate of the accuracy of the method can be obtained. The spectral density of the power flow between the two oscillators is also calculated, and it is shown that the energy in the system may flow backwards and forwards between the two oscillators extremely slowly, the energy

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of one oscillator building up while the other dies down and *vice versa*. This phenomenon is closely related to the well-known behaviour of the system in free vibration. However, although the illustrative examples of the use of the method relate only to a simple two-degree-of-freedom system, the method is perfectly general, and, as shown in the following development, applies equally to more complex multi-degree-of-freedom systems (for which exact solutions are not readily available).

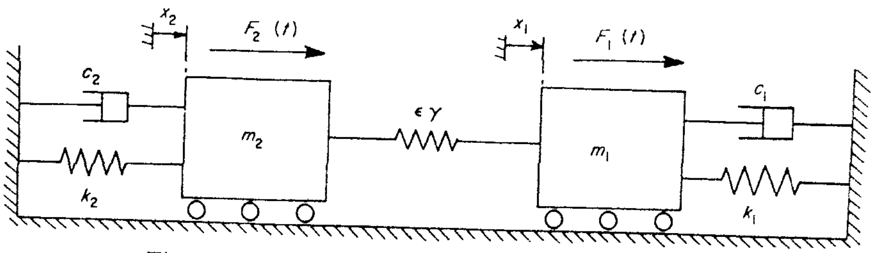

Figure 1. Mechanical model of two coupled oscillators.

GENERAL ANALYSIS

Consider a system of n uncoupled oscillators. Their equations of motion may be written

$$m_r\ddot{x}_r + c_r\dot{x}_r + k_r x_r = f_r(t), \qquad r = 1, 2, \ldots, n, \tag{1}$$

where, in mechanical engineering terms, m_r, c_r and k_r are the mass, damping and stiffness coefficients, and $f_r(t)$ is the excitation. The presence of small, linear, stiffness coupling terms modifies these equations to

$$m_r\ddot{x}_r + c_r\dot{x}_r + k_r x_r = f_r(t) + \sum_{\substack{s=1\\ s\neq r}}^{n} \epsilon\alpha_{rs}x_s, \qquad r = 1, 2, \ldots, n, \tag{2}$$

where $\epsilon \ll 1$. The α_{rs} coupling coefficients will not all be independent if the coupling is conservative, since they must then be derivable from a corresponding potential function V. This must have the form:

$$V = \sum_{r=1}^{n}\left[\tfrac{1}{2}k_r x_r^2 - \sum_{\substack{s=1\\ s\neq r}}^{n} \epsilon\beta_{rs}x_s x_r\right], \tag{3}$$

so that, to satisfy Lagrange's equations of motion the potential gradients are (3)

$$\frac{\partial V}{\partial x_r} = k_r x_r - \sum_{\substack{s=1\\ s\neq r}}^{n} \epsilon(\beta_{rs}+\beta_{sr})\,x_s. \tag{4}$$

By comparing equations (2) and (4), it can be seen that

$$\alpha_{rs} = \alpha_{sr} = (\beta_{rs}+\beta_{sr}). \tag{5}$$

Now the force exerted on the rth oscillator by all the other oscillators is

$$\sum_{\substack{s=1\\ s\neq r}}^{n} \epsilon\alpha_{rs}x_s \tag{6}$$

and consequently the rate of energy input to the rth oscillator from all the other oscillators is

$$\pi_r \text{ (say)} = \sum_{\substack{s=1\\ s\neq r}}^{n} \epsilon \alpha_{rs} x_s \dot{x}_r. \tag{7}$$

The mean power flow to the rth oscillator Π_r (say) is simply obtained by averaging equation (7) to give

$$\Pi_r = E[\pi_r] = \sum_{\substack{s=1\\ s\neq r}}^{n} \epsilon \alpha_{rs} E[x_s \dot{x}_r] \tag{8}$$

where the symbol E denotes the expectation or statistical average. Equation (8) may be rewritten in the following alternative form, which is more convenient for the later calculations:

$$\Pi_r = \sum_{\substack{s=1\\ s\neq r}}^{n} \epsilon \alpha_{rs} \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} E[x_s(t)\, x_r(t+\tau)]. \tag{9}$$

Since, for random excitation, the instantaneous power flow to the rth oscillator is itself a random variable, its spectral density can be calculated. The autocorrelation of the power flow is given, for stationary excitation, by

$$R_{\pi_r}(\tau) = E[\pi_r(t)\, \pi_r(t+\tau)], \tag{10}$$

which, after being combined with (7), can be written

$$R_{\pi_r}(\tau) = \epsilon^2 \sum_{\substack{q=1\\ q\neq r}}^{n} \sum_{\substack{s=1\\ s\neq r}}^{n} \alpha_{rq} \alpha_{rs} E[\dot{x}_r(t)\, \dot{x}_r(t+\tau)\, x_q(t)\, x_s(t+\tau)], \tag{11}$$

and the spectral density is the Fourier Transform of $R_{\pi_r}(\tau)$, defined as

$$S_{\pi_r}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{\pi_r}(\tau)\, \mathrm{e}^{-i\omega\tau}\, \mathrm{d}\tau. \tag{12}$$

The spectral density of π_r may therefore be calculated provided that $R_{\pi_r}(\tau)$ is known.

The problem of calculating the mean power flow Π_r and its spectral density $S_{\pi_r}(\omega)$ thus comes down to the calculation of the statistical moments in equations (9) and (11), and, in the case of $S_{\pi_r}(\omega)$, the integration of equation (12). In this analysis the moments required are calculated approximately by a perturbation technique.

It is assumed that the solution for $x_r(t)$ permits expansion in powers of the parameter ϵ:

$$x_r(t) = x_{r_0}(t) + \epsilon x_{r_1}(t) + \epsilon^2 x_{r_2}(t) + \ldots, \tag{13}$$

$$r = 1, 2, \ldots, n.$$

Substituting these expansions into equation (2) then gives, if the resulting equation is to be satisfied identically in ϵ, the following series of equations for successive approximations, with the forcing functions of higher order approximations made up of functions of the previous approximation:

$$m_r \ddot{x}_{r_0} + c_r \dot{x}_{r_0} + k_r x_{r_0} = f_r(t), \tag{14a}$$

$$m_r \ddot{x}_{r_1} + c_r \dot{x}_{r_1} + k_r x_{r_1} = \sum_{\substack{s=1\\ s\neq r}}^{n} \alpha_{rs} x_{s_0}, \tag{14b}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$r = 1, 2, \ldots, n.$$

From equation (14a) the solution for $x_{r_0}(t)$ is given by the superposition integral

$$x_{r_0}(t) = \int_0^\infty h_r(\theta) f_r(t-\theta)\,\mathrm{d}\theta, \tag{15}$$

where $h_r(\theta)$ is the response function of the rth oscillator to a unit impulse input $f_r(t) = \delta(\theta)$ when $\epsilon = 0$. Similarly, the response of $x_{r_1}(t)$ is, from equation (14b),

$$x_{r_1}(t) = \sum_{\substack{s=1\\ s\neq r}}^{n} \alpha_{rs} \int_0^\infty h_r(\theta)\, x_{s_0}(t-\theta)\,\mathrm{d}\theta. \tag{16}$$

The statistical moment required in equation (9) may be expanded, by using (13), to

$$E[x_s(t)\,x_r(t+\tau)] = E[x_{s_0}(t)\,x_{r_0}(t+\tau)] + \epsilon\{E[x_{s_1}(t)\,x_{r_0}(t+\tau)] + E[x_{s_0}(t)\,x_{r_1}(t+\tau)]\} + \mathrm{o}(\epsilon^2) \tag{17}$$

and, similarly, that in equation (11) to

$$E[\dot{x}_r(t)\,\dot{x}_r(t+\tau)\,x_q(t)\,x_s(t+\tau)] = E[\dot{x}_{r_0}(t)\,\dot{x}_{r_0}(t+\tau)\,x_{q_0}(t)\,x_{s_0}(t+\tau)] + \mathrm{o}(\epsilon). \tag{18}$$

Substituting the results (15) and (16) into equations (17) and (18) then gives final expressions for the required statistical moments, which may be extended by including higher order terms, to any accuracy desired. From (17)

$$\begin{aligned} E[x_s(t)\,x_r(t+\tau)] &= \int_0^\infty \mathrm{d}\theta_1 \int_0^\infty \mathrm{d}\theta_2\, h_s(\theta_1)\, h_r(\theta_2)\, E[f_s(t-\theta_1) f_r(t+\tau-\theta_2)] + \\ &+ \epsilon\Bigg\{ \sum_{\substack{q=1\\ q\neq s}}^{n} \alpha_{sq} \int_0^\infty \mathrm{d}\theta_1 \int_0^\infty \mathrm{d}\theta_2 \int_0^\infty \mathrm{d}\theta_3\, h_s(\theta_1)\, h_q(\theta_2)\, h_r(\theta_3)\, E[f_q(t-\theta_1-\theta_2) f_r(t+\tau-\theta_3)] + \\ &+ \sum_{\substack{q=1\\ q\neq r}}^{n} \alpha_{rq} \int_0^\infty \mathrm{d}\theta_1 \int_0^\infty \mathrm{d}\theta_2 \int_0^\infty \mathrm{d}\theta_3\, h_r(\theta_1)\, h_q(\theta_2)\, h_s(\theta_3)\, E[f_q(t+\tau-\theta_1-\theta_2) f_s(t-\theta_3)]\Bigg\} + \\ &+ \mathrm{o}(\epsilon^2). \end{aligned} \tag{19}$$

From (18)

$$\begin{aligned} E[\dot{x}_r(t)\,\dot{x}_r(t+\tau)\,x_q(t)\,x_s(t+\tau)] &= \int_0^\infty \mathrm{d}\theta_1 \int_0^\infty \mathrm{d}\theta_2 \int_0^\infty \mathrm{d}\theta_3 \int_0^\infty \mathrm{d}\theta_4\, h_r(\theta_1)\, h_r(\theta_2)\, h_q(\theta_3)\, h_s(\theta_4) \times \\ &\times E[f_r(t-\theta_1) f_r(t+\tau-\theta_2) f_q(t-\theta_3) f_s(t+\tau-\theta_4)] + \mathrm{o}(\epsilon). \end{aligned} \tag{20}$$

Provided that the excitation is fully specified, the moments on the right-hand sides of equations (19) and (20) are known. The integrations in (19) and (20) may then, in principle, be carried out, and the results substituted into equations (9) and (11) to give the mean power flow and the autocorrelation of the power flow.

A great simplification occurs if the excitation functions of the different oscillators are statistically independent of each other—an assumption of the so-called "energy approach" to structural vibration (4). In this case all the cross-correlations between the excitation functions are zero, and, for $r \neq s$, equation (19) becomes

$$E[x_s(t)\,x_r(t+\tau)]$$
$$= \epsilon\left\{\alpha_{sr}\int_0^\infty d\theta_1\int_0^\infty d\theta_2\int_0^\infty d\theta_3\, h_s(\theta_1)\,h_r(\theta_2)\,h_r(\theta_3)\,E[f_r(t-\theta_1-\theta_2)f_r(t+\tau-\theta_3)]+\right.$$
$$\left.+\alpha_{rs}\int_0^\infty d\theta_1\int_0^\infty d\theta_2\int_0^\infty d\theta_3\, h_r(\theta_1)\,h_s(\theta_2)\,h_s(\theta_3)\,E[f_s(t+\tau-\theta_1-\theta_2)f_s(t-\theta_3)]\right\}+o(\epsilon^2), \tag{21}$$

or, if this is combined with (5) and (16),

$$E[x_s(t)\,x_r(t+\tau)] = \epsilon\alpha_{rs}\int_0^\infty d\theta_1\{h_s(\theta_1)\,E[x_{r0}(t-\theta_1)\,x_{r0}(t+\tau)]+$$
$$+h_r(\theta_1)\,E[x_{s0}(t+\tau-\theta_1)\,x_{s0}(t)]\}+o(\epsilon^2). \tag{22}$$

If $R_{x_{r0}}(\tau)$ and $R_{x_{s0}}(\tau)$ denote the autocorrelation of the responses of the oscillators when $\epsilon=0$, equation (22) may be written as

$$E[x_s(t)\,x_r(t+\tau)] = \epsilon\alpha_{rs}\int_0^\infty d\theta_1\{h_s(\theta_1)\,R_{x_{r0}}(\tau+\theta_1)+$$
$$+h_r(\theta_1)\,R_{x_{s0}}(\tau-\theta_1)\}+o(\epsilon^2). \tag{23}$$

Differentiating this expression with respect to τ, making use of the fact that $R'_x(\tau)$ is an odd function of τ, and substituting into (9) gives

$$\Pi_r = \sum_{\substack{s=1\\ s\neq r}}^{n}\epsilon^2\alpha_{rs}^2\int_0^\infty [h_s(\theta)\,R'_{x_{r0}}(\theta)-h_r(\theta)\,R'_{x_{s0}}(\theta)]\,d\theta+o(\epsilon^3). \tag{24}$$

Equation (24) gives the mean rate of energy flow to the rth oscillator from all the other oscillators, correct to order ϵ^2 in the coupling parameter ϵ, for the case of uncorrelated excitation of each oscillator.

Equation (20) may also be greatly simplified if, as well as being statistically independent, the excitation functions are also Gaussian processes. In this case the fourth-order moment in (2) may be broken down into second-order moments (5):

$$E[f_r(t-\theta_1)f_r(t+\tau-\theta_2)f_q(t-\theta_3)f_s(t+\tau-\theta_4)]$$
$$= E[f_r(t-\theta_1)f_r(t+\tau-\theta_2)]\,E[f_q(t-\theta_3)f_s(t+\tau-\theta_4)]+$$
$$+E[f_r(t-\theta_1)f_q(t-\theta_3)]\,E[f_r(t+\tau-\theta_2)f_s(t+\tau-\theta_4)]+$$
$$+E[f_r(t-\theta_1)f_s(t+\tau-\theta_4)]\,E[f_r(t+\tau-\theta_2)f_q(t-\theta_3)]. \tag{25}$$

Since, from (11), $q\neq r$ and $s\neq r$, equation (25) reduces, if the excitation functions are statistically independent, to

$$E[f_r(t-\theta_1)f_r(t+\tau-\theta_2)f_q(t-\theta_3)f_s(t+\tau-\theta_4)]$$
$$= E[f_r(t-\theta_1)f_r(t+\tau-\theta_2)]\,E[f_s(t-\theta_3)f_s(t+\tau-\theta_4)] \tag{26}$$

and equation (20) may consequently be reduced to

$$E[\dot{x}_r(t)\,\dot{x}_r(t+\tau)\,x_s(t)\,x_s(t+\tau)] = R_{\dot{x}_{r0}}(\tau)\,R_{x_{s0}}(\tau)+o(\epsilon). \tag{27}$$

Substituting (27) into (11) gives

$$R_{\pi_r}(\tau) = \epsilon^2 R_{\dot{x}_{r0}}(\tau) \sum_{\substack{s=1\\ s\neq r}}^{n} \alpha_{rs}^2 R_{x_{s0}}(\tau) + o(\epsilon^3) \tag{28}$$

which, with (12), gives

$$\mathbb{S}_{\pi_r}(\omega) = \frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \left[R_{\dot{x}_{r0}}(\tau) \sum_{\substack{s=1\\ s\neq r}}^{n} \alpha_{rs}^2 R_{x_{s0}}(\tau)\, e^{-i\omega\tau} \right] d\tau + o(\epsilon^3) \tag{29}$$

as the final expression for the spectral density of the power flow to the rth oscillator. It applies for the case when the excitation functions of the oscillators are mutually independent and Gaussian. $R_{\dot{x}_{r0}}(\tau)$ is the autocorrelation of the derivative of the response of the rth oscillator when $\epsilon = 0$, and $R_{x_{s0}}(\tau)$ is the autocorrelation of the sth oscillator when $\epsilon = 0$. The answer is correct to order ϵ^2 in the coupling parameter ϵ.

MEAN POWER FLOW FOR WHITE NOISE EXCITATION

For the special case of white noise excitation, the expression for mean power flow, equation (24), may be further simplified. If the natural frequency and damping ratio of the rth oscillator are Ω_r and β_r, where

$$\Omega_r^2 = \frac{k_r}{m_r}, \quad 2\beta_r \Omega_r = \frac{c_r}{m_r}, \tag{30}$$

then the autocorrelation of the response to white noise, of spectral density S_r, is (6)

$$\underset{\theta>0}{R_{x_{r0}}(\theta)} = \frac{\pi S_r}{2m_r^2 \beta_r \Omega_r^3} e^{-\beta_r \Omega_r \theta} \left\{ \cos\sqrt{1-\beta_r^2}\,\Omega_r \theta + \frac{\beta_r}{\sqrt{1-\beta_r^2}} \sin\sqrt{1-\beta_r^2}\,\Omega_r \theta \right\} \tag{31}$$

so that

$$\underset{\theta>0}{R'_{x_{r0}}(\theta)} = -\frac{\pi S_r}{2m_r^2 \beta_r \sqrt{1-\beta_r^2}\,\Omega_r^2} e^{-\beta_r \Omega_r \theta} \{\sin\sqrt{1-\beta_r^2}\,\Omega_r \theta\}. \tag{32}$$

The impulse response function is, from equation (14a) (7),

$$\underset{\theta>0}{h_r(\theta)} = \frac{1}{m_r \sqrt{1-\beta_r^2}\,\Omega_r} e^{-\beta_r \Omega_r \theta} \{\sin\sqrt{1-\beta_r^2}\,\Omega_r \theta\}. \tag{33}$$

Substituting (32) and (33) into (24) gives, for the mean power flow,

$$\Pi_r = \sum_{\substack{s=1\\ s\neq r}}^{n} \epsilon^2 \alpha_{rs}^2 \int_0^{\infty} h_r(\theta) h_s(\theta) \left[\frac{R'_{x_{r0}}(\theta)}{h_r(\theta)} - \frac{R'_{x_{s0}}(\theta)}{h_s(\theta)} \right] d\theta + o(\epsilon^3), \tag{34}$$

$$= \sum_{\substack{s=1\\ s\neq r}}^{n} \epsilon^2 \alpha_{rs}^2 \int_0^{\infty} h_r(\theta) h_s(\theta) \left[\frac{\pi S_s}{2m_s \beta_s \Omega_s} - \frac{\pi S_r}{2m_r \beta_r \Omega_r} \right] d\theta + o(\epsilon^3). \tag{35}$$

The term $(\pi S_r)/(2m_r \beta_r \Omega_r)$ is the mean energy of the rth oscillator, U_{r0} say, when $\epsilon = 0$. Since the total mean energy is twice the mean kinetic energy,

$$U_{r0} = m_r E[\dot{x}_{r0}^2]. \tag{36}$$

The right-hand side of (36) may be obtained most conveniently from (14a) by multiplying through by $\dot{x}_{r0}$ and then averaging to give

$$c_r E[\dot{x}_{r0}^2] = E[f_r(t)\, \dot{x}_{r0}(t)] \tag{37}$$

since $E[\ddot{x}_{r0}\dot{x}_{r0}] = E[x_{r0}\dot{x}_{r0}] = 0$ for any stationary process.† From (14a)

$$\dot{x}_{r0}(t) = \int_0^\infty h_r(\theta) f_r(t-\theta)\,\mathrm{d}\theta, \tag{38}$$

and substituting this into the right-hand side of (37) gives

$$c_r E[\dot{x}_{r0}^2] = \int_0^\infty h_r(\theta)\, E[f_r(t) f_r(t-\theta)]\,\mathrm{d}\theta. \tag{39}$$

For white noise excitation $E[f_r(t)f_r(t-\theta)] = 2\pi S_r \delta(\theta)$, so that (39) becomes

$$cE[\dot{x}_{r0}^2] = 2\pi S_r \int_0^\infty h_r(\theta)\,\delta(\theta)\,\mathrm{d}\theta, \tag{40}$$

$$= \frac{\pi S_r}{m_r}, \tag{41}$$

since $h_r(0-) = 0$, $h_r(0+) = 1/m_r$. Substituting (41) into (36) then gives, for the mean energy of the rth oscillator,

$$U_{r0} = \frac{\pi S_r}{c_r} = \frac{\pi S_r}{2m_r \beta_r \Omega_r}. \tag{42}$$

Putting (42) into (35) gives

$$\Pi_r = \sum_{\substack{s=1 \\ s\neq r}}^{n} \epsilon^2 \alpha_{rs}^2 [U_{s0} - U_{r0}] \int_0^\infty h_r(\theta)\, h_s(\theta)\,\mathrm{d}\theta + o(\epsilon^3). \tag{43}$$

The rate of energy flow from the sth to the rth oscillator is therefore directly proportional to the difference between the mean energies of the two oscillators. The integral in (43) may be evaluated by straightforward methods after substituting for $h_r(\theta)$ and $h_s(\theta)$ from (33). The result is

$$\Pi_r = \sum_{\substack{s=1 \\ s\neq r}}^{n} \frac{\epsilon^2 \alpha_{rs}^2}{m_r m_s} (U_{s0} - U_{r0}) \left[\frac{2(\beta_r \Omega_r + \beta_s \Omega_s)}{\{(\beta_r \Omega_r + \beta_s \Omega_s)^2 + (p_r - p_s)^2\}\{(\beta_r \Omega_r + \beta_s \Omega_s)^2 + (p_r + p_s)^2\}} \right] + $$
$$+ o(\epsilon^3), \tag{44}$$

where
$$p_r = \Omega_r \sqrt{1-\beta_r^2}, \quad p_s = \Omega_s \sqrt{1-\beta_s^2}.$$

Equation (44) gives, correct to order ϵ^2, the mean power flow to the rth oscillator from all the other oscillators. It applies when the excitation of each oscillator is an independent source of stationary, white noise. Since the term in square brackets in (44) is always positive, energy always flows from higher to lower energy oscillators. Furthermore, when two lightly damped oscillators are approximately in tune with each other, $p_r \simeq p_s$, then the denominator inside the square brackets will be small, and the rate of energy transfer increases. To the accuracy of this calculation, therefore, oscillators prefer to share energy, equipartition of energy being a state of stable equilibrium, and their resistance to energy

† These results are easily proved. Consider $E[x\dot{x}]$. For stationary processes,

$$E[x\dot{x}] = \frac{d}{d\tau} E[x(t)x(t+\tau)]\bigg|_{\tau=0} = \frac{d}{d\tau} E[x(t-\tau)x(t)]\bigg|_{\tau=0} = -E[\dot{x}x],$$

so that $E[x\dot{x}] = 0$. Similarly, $E[\dot{x}\ddot{x}] = 0$.

transfer depends on their natural frequency ratio, going through a minimum when $p_r \simeq p_s$.†

EXAMPLE OF TWO COUPLED OSCILLATORS

To illustrate the application of the above results, consider the mechanical model of two stiffness coupled oscillators shown in Figure 1. If the displacements from static equilibrium of the two masses are x_1 and x_2 and if the external forces $F_1(t)$ and $F_2(t)$ are applied to the masses, then the equations of motion are

$$\begin{aligned} m_1 \ddot{x}_1 + c_1 \dot{x}_1 + (k_1 + \epsilon\gamma) x_1 &= F_1(t) + \epsilon\gamma x_2, \\ m_2 \ddot{x}_2 + c_2 \dot{x}_2 + (k_2 + \epsilon\gamma) x_2 &= F_2(t) + \epsilon\gamma x_1, \end{aligned} \tag{45}$$

which may alternatively be written as

$$\begin{aligned} \ddot{x}_1 + 2\beta_1 \Omega_1 \dot{x}_1 + \Omega_1^2 x_1 &= \frac{F_1(t)}{m_1} + \frac{\epsilon\gamma}{m_1} x_2, \\ \ddot{x}_2 + 2\beta_2 \Omega_2 \dot{x}_2 + \Omega_2^2 x_2 &= \frac{F_2(t)}{m_2} + \frac{\epsilon\gamma}{m_2} x_1, \end{aligned} \tag{46}$$

where Ω_1 and β_1 are, respectively, the "blocked" natural frequency and damping ratio for oscillator 1 with oscillator 2 clamped, as defined by the equations

$$\Omega_1^2 = \frac{k_1 + \epsilon\gamma}{m_1}, \quad \beta_1 = \frac{c_1}{2\sqrt{(k_1 + \epsilon\gamma) m_1}}, \tag{47}$$

and, similarly, Ω_2 and β_2 are the "blocked" natural frequency and damping ratio for oscillator 2 with oscillator 1 clamped, as defined by

$$\Omega_2^2 = \frac{k_2 + \epsilon\gamma}{m_2}, \quad \beta_2 = \frac{c_2}{2\sqrt{(k_2 + \epsilon\gamma) m_2}}. \tag{48}$$

It will be assumed that $F_1(t)$ and $F_2(t)$ are two *independent sources of stationary*, *Gaussian*, white noise random excitation.

Consider the case when only one oscillator is excited. The energy of the other then arises solely from the power flowing between the two. If $F_2(t) = 0$, then, from (44), the power flow to the second oscillator is

$$\Pi_2 = \frac{e^2\gamma^2}{m_1 m_2} U \cdot \left[\frac{2(\beta_1 \Omega_1 + \beta_2 \Omega_2)}{\{(\beta_1 \Omega_1 + \beta_2 \Omega_2)^2 + (p_1 - p_2)^2\}\{(\beta_1 \Omega_1 + \beta_2 \Omega_2)^2 + (p_1 + p_2)^2\}}\right]. \tag{49}$$

This power input is dissipated by the damping of the oscillator, and the mean rate of power dissipation is

$$c_2 E[\dot{x}_2^2] = 2m_2 \beta_2 \Omega_2 E[\dot{x}_2^2]. \tag{50}$$

Equating (49) and (50) allows the mean kinetic energy of the oscillator to be written as

$$\begin{aligned} T_1 = \tfrac{1}{2} m_2 E[\dot{x}_2^2] &= \epsilon^2 \gamma^2 \frac{U_{10}}{2m_1 m_2 \beta_2 \Omega_2} \times \\ &\times \left[\frac{(\beta_1 \Omega_1 + \beta_2 \Omega_2)}{\{(\beta_1 \Omega_1 + \beta_2 \Omega_2)^2 + (p_1 - p_2)^2\}\{(\beta_1 \Omega_1 + \beta_2 \Omega_2)^2 + (p_1 + p_2)^2\}}\right]. \end{aligned} \tag{51}$$

† Subsequent to the preparation of this paper, it has been found that this is an exact result for two coupled oscillators (8). In the case of *strong coupling, the energy in the coupling element itself* must be taken into account and divided up between the oscillators in a particular way. It has not, however, so far been possible to prove that the result is exact for more than two oscillators.

If the "blocked" kinetic energy of the first oscillator is defined as its energy when the second oscillator is clamped, so that

$$T_{1B} = \tfrac{1}{2} m_1 E[\dot{x}_{10}^2] = \tfrac{1}{2} U_{10}, \tag{52}$$

equation (51) can be rewritten in non-dimensional form as

$$\frac{T_2}{T_{1B}} = \mathbb{C}\left[\frac{\left\{1+\dfrac{\beta_1\Omega_1}{\beta_2\Omega_2}\right\}\left(\dfrac{\Omega_2}{\Omega_1}\right)^2}{\left\{\left(\beta_1+\dfrac{\beta_2\Omega_2}{\Omega_1}\right)^2+\left(\dfrac{p_1-p_2}{\Omega_1}\right)^2\right\}\left\{\left(\beta_1+\dfrac{\beta_2\Omega_2}{\Omega_1}\right)^2+\left(\dfrac{p_1+p_2}{\Omega_1}\right)^2\right\}}\right], \tag{53}$$

where the coupling coefficient $\mathbb{C}$ is defined as

$$\mathbb{C} = \frac{\epsilon^2\gamma^2}{m_1 m_2 \Omega_1^2 \Omega_2^2} = \frac{\epsilon^2\gamma^2}{(k_1+\epsilon\gamma)(k_2+\epsilon\gamma)} \simeq \frac{\epsilon^2\gamma^2}{k_1 k_2}. \tag{54}$$

In Figure 2, T_2/T_{1B} is shown plotted against the ratio of the natural frequencies Ω_2/Ω_1 for the case $\mathbb{C} = 10^{-5}$ and $\beta_1 = \beta_2 = 0{\cdot}01$. As explained in the Appendix, T_2/T_{1B} can, in this case, be calculated exactly fairly easily, and the exact result is also shown in the inset view in Figure 2. There is good agreement between the approximate and exact results except

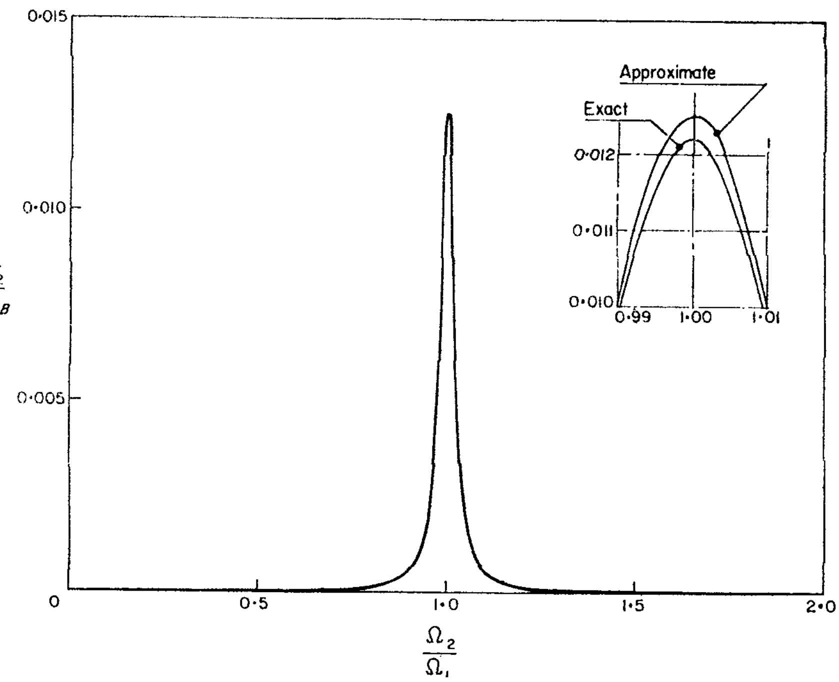

Figure 2. Mean kinetic energy (T_2) of the second oscillator in Figure 1, as a function of the ratio of the oscillator natural frequencies (Ω_2/Ω_1), for the case when only the first oscillator is excited with white noise. (T_{1B} is the blocked kinetic energy of the first oscillator under the same excitation, $\mathbb{C} = \dfrac{\epsilon^2\gamma^2}{m_1 m_2 \Omega_1^2 \Omega_2^2} = 10^{-5}$, $\beta_1 = \beta_2 = 0{\cdot}01$.)

where the two natural frequencies are almost exactly the same, $\Omega_2 \simeq \Omega_1$. There is then considerable excitation of the second oscillator, which causes a corresponding reduction in the energy of the first. This reduction is not allowed for in the approximate theory.

The energy of the first oscillator may easily be calculated once that of the second is known. By the conservation of energy, the mean rate of energy dissipation of both oscillators,

$$c_1 E[\dot{x}_1^2] + c_2 E[\dot{x}_2^2], \tag{55}$$

must be equal to the mean rate of energy input from the external force $F_1(t)$, which is

$$E[\dot{x}_1(t) F_1(t)]. \tag{56}$$

For white noise excitation of spectral density S_1, (56) may be evaluated along the lines of equations (37)–(41) to find

$$E[\dot{x}_1(t) F_1(t)] = \pi S_1 \dot{\mathscr{H}}_1(t=0+), \tag{57}$$

where $\mathscr{H}_1(t)$ is now the response $\dot{x}_1(t)$ as a result of a unit impulse input $F_1(t) = \delta(t)$. [$\dot{\mathscr{H}}_1(t)$ in general differs from $h_1(t)$, which is the impulse response for $\dot{x}_1(t)$ with the second oscillator clamped.] From the first of (45)

$$\dot{\mathscr{H}}_1(t=0+) = \frac{1}{m_1}, \tag{58}$$

giving

$$E[\dot{x}_1(t) F_1(t)] = \frac{\pi S_1}{m_1}. \tag{59}$$

Equating (55) and (59) for the conservation of energy then gives

$$c_1 E[\dot{x}_1^2] + c_2 E[\dot{x}_2^2] = \frac{\pi S_1}{m_1}, \tag{60}$$

which, after some rearrangement, may be written as

$$T_1 + \frac{\beta_2 \Omega_2}{\beta_1 \Omega_1} T_2 = \frac{\pi S_1}{4 m_1 \beta_1 \Omega_1}, \tag{61}$$

or, using (42) and (52),

$$\frac{T_1}{T_{1B}} = 1 - \frac{\beta_2 \Omega_2}{\beta_1 \Omega_1} \frac{T_2}{T_{1B}}. \tag{62}$$

Equation (62) is an exact result which is independent of the magnitude of the coupling.

In Figure 3 the energies of both oscillators are shown for a more strongly coupled system than that of Figure 2. In this case $\mathbb{C} = 10^{-4}$ instead of 10^{-5}, while $\beta_1 = \beta_2 = 0{\cdot}01$ as before. There is a marked reduction in the energy of the first oscillator, as it loses energy to the second when their natural frequencies are approximately the same. For the approximate theory to be accurate, the coupling must be sufficiently small that the loading effect of one oscillator on the other does not cause the kinetic energy difference $(T_1 - T_2)$ to differ significantly from $(T_{1B} - T_{2B})$, where T_{1B} and T_{2B} are the "blocked" values for the kinetic energies. In many practical cases in which the aim is to prevent the transmission of noise energy from one structure to another this is often a realistic assumption.

It is interesting to note that the approximate result obtained here for the mean power flow between two oscillators is identical with that obtained by Lyon and Maidanik (1) using a different method of approximation. The discrepancy between the approximate and exact results illustrated above is thus also a feature of the earlier analysis. The advantages of the present method (as an additional tool in the study of the random vibration of coupled oscillator systems) lie in its usefulness in allowing the approximations made to remain more readily apparent, in its capability of being extended to second and higher order approximations if greater accuracy is needed, and in its applications to the calculation of statistics more complicated than those of mean power flow. As an illustration of the last, the spectrum of the power flow between the two oscillators of Figure 1 is calculated below for the case when both oscillators are excited.

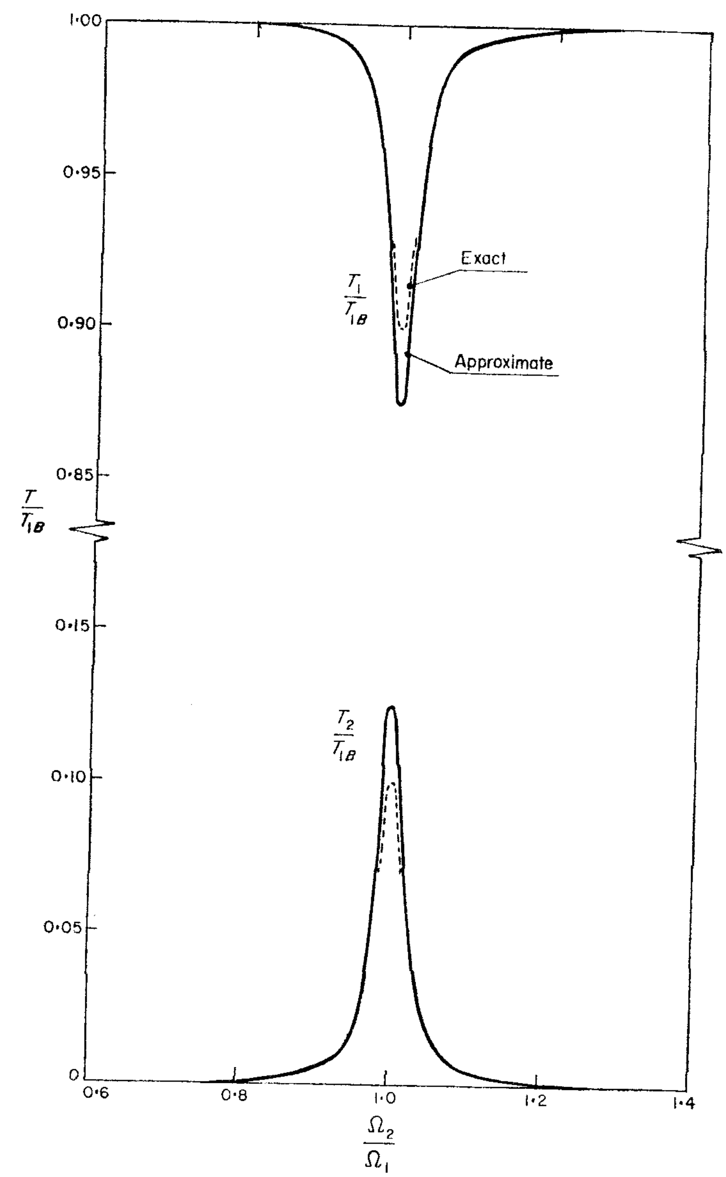

Figure 3. Mean kinetic energies of both oscillators when only the first is excited with white noise. ($\mathbb{C} = 10^{-4}$, $\beta_1 = \beta_2 = 0{\cdot}01$.)

From the general result of equation (29), since both $R_{\dot{x}_{1_0}}(\tau)$ and $R_{x_{2_0}}(\tau)$ are even functions of τ, the spectral density of the power flow may be written as

$$\mathbb{S}_{\pi_1}(\omega) = \frac{\epsilon^2\gamma^2}{\pi} \int_0^\infty R_{x_{2_0}}(\tau)\, R_{\dot{x}_{1_0}}(\tau) \cos\omega\tau \,\mathrm{d}\tau. \tag{63}$$

For stationary white noise excitation, from the result that

$$R_{\dot{x}_{1_0}}(\tau) = E[\dot{x}_{1_0}(t)\,\dot{x}_{1_0}(t+\tau)] = \frac{\mathrm{d}}{\mathrm{d}\tau}E[\dot{x}_{1_0}(t)\,x_{1_0}(t+\tau)], \tag{64}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\tau}E[\dot{x}_{1_0}(t-\tau)\,x_{1_0}(t)], \tag{65}$$

$$= -\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}E[x_{1_0}(t-\tau)\,x_{1_0}(t)], \tag{66}$$

$$= -R''_{x_{1_0}}(\tau), \tag{67}$$

and from the expressions for $R_{x_{1_0}}(\tau)$, $R_{x_{2_0}}(\tau)$ in equation (31,) then

$$\mathbb{S}_{\pi_1}(\omega) = \epsilon^2\gamma^2\frac{\pi S_1 S_2}{4m_1^2 m_2^2\beta_1\beta_2\Omega_1\Omega_2^3}\int_0^\infty \mathrm{e}^{-(\beta_1\Omega_1+\beta_2\Omega_2)\tau}\times$$

$$\times\left\{\cos p_1\tau - \frac{\beta_1}{\sqrt{1-\beta_1^2}}\sin p_1\tau\right\}\left\{\cos p_2\tau + \frac{\beta_2}{\sqrt{1-\beta_2^2}}\sin p_2\tau\right\}\cos\omega\tau\,\mathrm{d}\tau, \tag{68}$$

and multiplying out and evaluating the integrals gives

$$\mathbb{S}_{\pi_1}(\omega) = \mathbb{C}T_{1\mathrm{B}}T_{2\mathrm{B}}\frac{\Omega_1}{\pi}\times$$

$$\times\Bigg[\left(1-\frac{\beta_1\beta_2}{\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}\right)\times$$

$$\times\left\{\frac{\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}}{\left(\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}\right)^2+\left(\frac{p_1-p_2-\omega}{\Omega_1}\right)^2}+\frac{\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}}{\left(\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}\right)^2+\left(\frac{p_1-p_2+\omega}{\Omega_1}\right)^2}\right\}+$$

$$+\left(1+\frac{\beta_1\beta_2}{\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}\right)\times$$

$$\times\left\{\frac{\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}}{\left(\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}\right)^2+\left(\frac{p_1+p_2-\omega}{\Omega_1}\right)^2}+\frac{\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}}{\left(\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}\right)^2+\left(\frac{p_1+p_2+\omega}{\Omega_1}\right)^2}\right\}+$$

$$+\left(\frac{\beta_2}{\sqrt{1-\beta_2^2}}-\frac{\beta_1}{\sqrt{1-\beta_1^2}}\right)\times$$

$$\times\left\{\frac{\frac{p_1+p_2+\omega}{\Omega_1}}{\left(\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}\right)^2+\left(\frac{p_1+p_2+\omega}{\Omega_1}\right)^2}+\frac{\frac{p_1+p_2-\omega}{\Omega_1}}{\left(\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}\right)^2+\left(\frac{p_1+p_2-\omega}{\Omega_1}\right)^2}\right\}+$$

$$+\left(\frac{\beta_2}{\sqrt{1-\beta_2^2}}+\frac{\beta_1}{\sqrt{1-\beta_1^2}}\right)\times$$

$$\times\left\{\frac{\frac{p_2-p_1+\omega}{\Omega_1}}{\left(\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}\right)^2+\left(\frac{p_2-p_1+\omega}{\Omega_1}\right)^2}+\frac{\frac{p_2-p_1-\omega}{\Omega_1}}{\left(\beta_1+\beta_2\frac{\Omega_2}{\Omega_1}\right)^2+\left(\frac{p_2-p_1-\omega}{\Omega_1}\right)^2}\right\}\Bigg],$$

where again $p_1 = \Omega_1\sqrt{1-\beta_1^2}$, $p_2 = \Omega_2\sqrt{1-\beta_2^2}$. (69)

The spectral density $\mathbb{S}_{\pi_1}(\omega) = \Omega_1 \, \mathbb{S}_{\pi_1}(\omega/\Omega_1)$ is shown plotted against ω/Ω_1 in Figure 4 for the case $\Omega_2/\Omega_1 = 0{\cdot}9$ and $\beta_1 = \beta_2 = 0{\cdot}01$. There are "resonance-like" peaks in the spectrum at $\omega \simeq |p_2 - p_1|$ and at $\omega \simeq |p_2 + p_1|$. The power flow thus has a low frequency component, whose centre frequency approaches zero when $p_2 \rightarrow p_1$. There is then a very slow cycling of energy backwards and forwards between the two oscillators. Even though their mean energies may be the same, so that no net power flow occurs, there is nevertheless a steady flow of energy backwards and forwards so that the energy of one oscillator slowly builds up while the other dies down and *vice versa*. This phenomenon is, of course, closely related to the well-known performance of the system in free vibration (9).

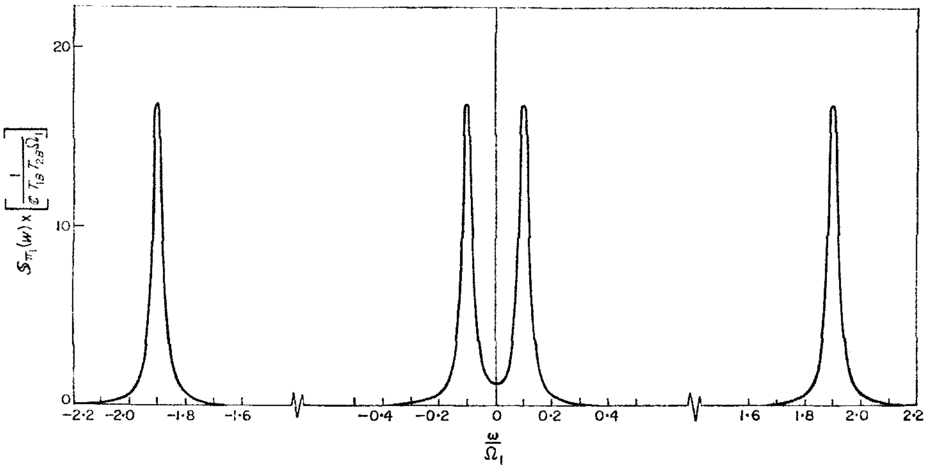

Figure 4. Mean square spectral density of the power flow to oscillator 1, from oscillator 2, $\mathbb{S}_{\pi_1}(w)$ when both are excited by independent sources of Gaussian white noise. (T_{1B} and T_{2B} are the blocked mean kinetic energies of the two oscillators under the same excitation, $\Omega_2/\Omega_1 = 0{\cdot}9$, $\beta_1 = \beta_2 = 0{\cdot}01$.)

CONCLUSION

An approximate perturbation method has been developed to calculate the statistics of the power flow between randomly excited, weakly coupled oscillators. This has been applied to calculate the mean power flow between two stiffness coupled oscillators. Although there is also an exact solution for this problem, for more complicated cases, involving more oscillators or higher order statistics, the exact solution is prohibitively involved, whereas the approximate method developed here is directly applicable. As an example, the spectral density of the power flow between two coupled oscillators has been calculated. The method shows good promise as an additional tool for the study of noise transmission in structures, and it is hoped that later publications will deal with this development. The important general results that have been derived here by using the method are that, subject to assumptions and approximations explained in the paper, (a) energy always flows in the direction of the energy gradient, (b) the mean power flow between oscillators is only significant when they have approximately the same natural frequencies, and (c) that energy can cycle slowly backwards and forwards between two oscillators at a low frequency, of the order of the difference between the two oscillator natural frequencies.

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APPENDIX: EXACT CALCULATION OF THE MEAN KINETIC ENERGY OF A COUPLED OSCILLATOR

For the system shown in Figure 1, when $F_2(t) = 0$,

$$\begin{aligned} \ddot{x}_1 + 2\beta_1 \Omega_1 \dot{x}_1 + \Omega_1^2 x_1 &= \frac{F_1(t)}{m_1} + \epsilon \frac{\gamma}{m_1} x_2, \\ \ddot{x}_2 + 2\beta_2 \Omega_2 \dot{x}_2 + \Omega_2^2 x_2 &= \epsilon \frac{\gamma}{m_2} x_1, \end{aligned} \tag{70}$$

where Ω_1, Ω_2, β_1 and β_2 are defined by equations (47) and (48). Eliminating x_1 between equations (70) gives

$$\ddddot{x}_2 + A\dddot{x}_2 + B\ddot{x}_2 + C\dot{x}_2 + Dx_2 = \epsilon \frac{\gamma}{m_2} \frac{F_1(t)}{m_1}, \tag{71}$$

where

$$\begin{aligned} A &= 2(\beta_1 \Omega_1 + \beta_2 \Omega_2), \\ B &= (\Omega_1^2 + \Omega_2^2 + 4\beta_1 \beta_2 \Omega_1 \Omega_2), \\ C &= 2\Omega_1 \Omega_2 (\beta_1 \Omega_2 + \beta_2 \Omega_1), \\ D &= \Omega_1^2 \Omega_2^2 - \frac{\epsilon^2 \gamma^2}{m_1 m_2}. \end{aligned} \tag{72}$$

The response $x_2(t)$ to a harmonic input

$$F_1(t) = e^{i\omega t} \tag{73}$$

is given by

$$x_2(t) = H(\omega)\, e^{i\omega t} \tag{74}$$

where $H(\omega)$ is the complex frequency response function for x_2. Substituting (73) and (74) into (71) gives

$$H(\omega) = \frac{\dfrac{\epsilon\gamma}{m_1 m_2}}{\omega^4 - i\omega^3 A - \omega^2 B + i\omega C + D}. \tag{75}$$

The mean square velocity $E[\dot{x}_2^2]$ is then given by (10)

$$E[\dot{x}_2^2] = \int_{-\infty}^{\infty} |i\omega H(\omega)|^2 S_{F_1}(\omega)\,\mathrm{d}\omega \tag{76}$$

where $S_{F_1}(\omega)$ is the mean square spectral density of the input force $F_1(t)$. For white noise excitation, $S_{F_1}(\omega) = S_1$, the integral in (76) may be evaluated by standard methods, and the results for such integrals have been tabulated (11) giving

$$E[\dot{x}_2^2] = \frac{\dfrac{\epsilon^2\gamma^2}{m_1 m_2} A\pi S_1}{C(AB-C)-DA^2}. \tag{77}$$

The mean kinetic energy of the second oscillator is consequently

$$T_2 = \tfrac{1}{2} m_2 E[\dot{x}_2^2] = \epsilon^2\gamma^2 \frac{\pi S_1}{2m_2 m_1^2}\left[\frac{A}{C(AB-C)-DA^2}\right] \tag{78}$$

which, after a good deal of algebra, reduces to

$$\frac{T_2}{T_{1B}} = \frac{\mathbb{C}\left(\dfrac{\Omega_2}{\Omega_1}\right)^2\left(1+\dfrac{\beta_1\Omega_1}{\beta_2\Omega_2}\right)}{\left\{\left(1-\dfrac{\Omega_2^2}{\Omega_1^2}\right)^2 + 4\beta_1\beta_2\dfrac{\Omega_2}{\Omega_1}\left(1+\dfrac{\beta_2\Omega_2}{\beta_1\Omega_1}\right)\left(1+\dfrac{\beta_1\Omega_2}{\beta_2\Omega_1}\right)\right\} + \mathbb{C}\left\{\dfrac{\beta_1\Omega_2}{\beta_2\Omega_1}\left(1+\dfrac{\beta_2\Omega_2}{\beta_1\Omega_1}\right)^2\right\}} \tag{79}$$

with, as before,

$$\mathbb{C} = \frac{\epsilon^2\gamma^2}{m_1 m_2 \Omega_1^2 \Omega_2^2} \simeq \frac{\epsilon^2\gamma^2}{k_1 k_2}. \tag{54}$$

This may be compared with the simpler approximate expression (53) for the same result. Equation (79) has been used to obtain the exact results shown in Figures 2 and 3.